CRANBROOK SCHOOL

YEAR 12 MATHEMATICS – EXTENSION 1

Term 3 2003 **Time : 2 h / CJL, HRK and SKB**

All questions are of equal value.
All necessary working should be shown in every question.
Full marks may not be awarded if work is careless or badly arranged.
Approved silent calculators may be used.
Submit your work in seven 4 Page Booklets.

1. **(12marks) (Begin a 4 page booklet.)** **CJL**

(a) Consider the parabola $x^2 = 12y$.

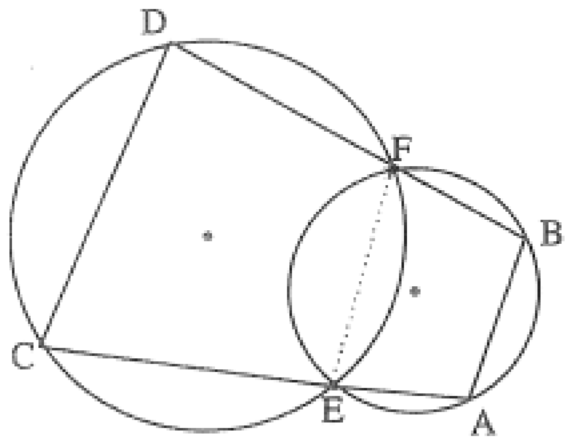
(i) Show that $P(6p, 3p^2)$ lies on the parabola. 1

(ii) Find the coordinates of the midpoint of the chord joining $P(6p, 3p^2)$ and $Q(6q, 3q^2)$. 1

(iii) The chord PQ has a gradient of 1. Prove that $p + q = 2$. 1

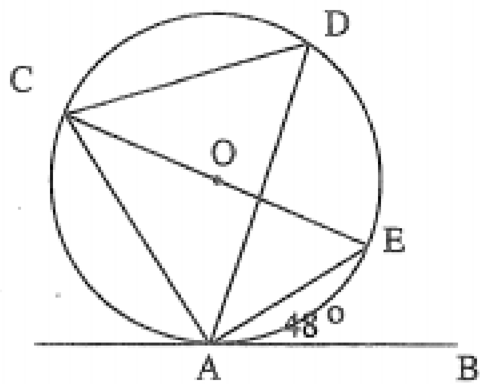
(iv) Hence find the equation of the locus of midpoints of chords with a gradient of 1, noting any specific restrictions on this locus. 3

(b)



Two circles intersect at E and F. AEC and BFD are straight lines. Copy the diagram and prove that AB is parallel to CD. 3

(c)



AB is a tangent and CE is a diameter to a circle of centre O. Angle BAE is 48^0 and D lies on the circumference as shown in the diagram.

(i) Copy the diagram and find the size of angle ACE, giving reasons. 1

(ii) Find the size of angle ADC. Justify your answer. 2

2. **(12marks) (Begin a 4 page booklet.)** **CJL**

(a) Find the general solution of: $\cos 2x + \sqrt{3}\sin 2x = 1$ 4

(b) Find $\int \sin^2 6x \, dx$ 2

(c) Using the substitution $x = 5\sec\theta$, find the indefinite integral

$$\int \frac{dx}{x\sqrt{x^2 - 25}}$$ 3

(d) Find the exact value of $\displaystyle\int_0^{\frac{1}{2}\ln 3} \frac{e^x}{1+e^{2x}}\,dx$,

using the substitution $u = e^x$. 3

3. (12marks) (Begin a 4 page booklet.) HRK

(a) (i) Use the method of mathematical induction to show that if x is a positive integer then $(1+x)^n - 1$ is divisible by x for all positive integers $n \geq 1$. 4

(ii) Factorise $12^n - 4^n - 3^n + 1$. Without using the method of mathematical induction again, use the result of part (i) to deduce that $12^n - 4^n - 3^n + 1$ is divisible by 6 for all positive integers $n \geq 1$. 2

(b) Consider the function $f(x) = \dfrac{x}{4-x^2}$.

(i) Find the domain of the function. 1

(ii) Show that the function is an odd function. 1

(iii) Show that the function is increasing throughout its domain. 1

(iv) Sketch the graph of the function showing clearly the coordinates of any points of intersection with the x axis or the y axis and the equations of any asymptotes. 2

(v) Use the graph of the function to explain whether or not the inverse function exists. 1

4. (12marks) (Begin a 4 page booklet.) HRK

(a) A monic cubic polynomial when divided by x^2+4 leaves a remainder of $x+8$ and when divided by x leaves a remainder of –4. Find the polynomial in the form ax^3+bx^2+cx+d. 6

(b) (i) By considering the graph of $y = e^x$, show that the equation $e^x + x + 1 = 0$ has only one real root and that this root is negative. 2

(ii) Taking $x = -1.5$ as a first approximation to this root, use one application of Newton's Method to find a better approximation correct to 2 decimal places. 4

5. **(12marks) (Begin a 4 page booklet.)** **SKB**

(a) Consider the function $f(x) = 1 + \frac{2}{x-3}$ for $x > 3$.

(i) Find the inverse function $f^{-1}(x)$. 2

(ii) Hence sketch $y = f^{-1}(x)$. 2

(b) Find the exact value of $\sin[\tan^{-1}(\frac{3}{2}) + \cos^{-1}(\frac{2}{3})]$ 4

(c) The portion of the curve $(16 - x^2)y^4 = 2$ (for which y is positive) is rotated about the x-axis from $x = 0$ to $x = 2$. Find the exact volume of revolution generated. 4

6. **(12marks) (Begin a 4 page booklet.)** **SKB**

(a) Solve for x : $\frac{x^2 - 5x}{4 - x} \leq -3$ 4

(b) When the interval joining the points (-5,6) and (-2,3) is divided externally in the ratio m:n the point of division is (4,-3). Find m:n. 4

(c) A particle moving in simple harmonic motion has a velocity $v\,\text{ms}^{-1}$ given by $v^2 = 15 + 2x - x^2$, where x is the displacement in metres.

(i) Find the end points of the motion. 2

(ii) Find the acceleration when the particle is at x = -2. 2

7. (12marks) (Begin a 4 page booklet.) **SKB**

(a) In a particular equatorial African swamp a colony of tsetse flies increases its population according to the differential equation $\frac{dP}{dt} = k(P - 10000)$, where k is the growth rate of the colony.
Initially there were 15000 tsetse flies and after 6 months there were 25000 tsetse flies.

(i) Show that $P = 10000 + P_o e^{kt}$ is a solution of this differential equation. 2

(ii) Determine the growth rate k and P_0 in exact form. 2

(iii) Determine the number of tsetse flies after 1 year. 2

(b) A missile is launched upwards from a submarine 40 m below sea level at an angle θ to the horizontal with a speed of $30\,\text{ms}^{-1}$.
After reaching its maximum height after $\frac{3\sqrt{3}}{2}$s the missile strikes a frigate located 3km away in a horizontal direction with respect to the sea level axis. Assuming that the acceleration due to gravity, g is 10 ms^{-2} and neglecting any air or water resistance :

(i) Show that the parametric equations of motion are given by: $x = 30t\cos\theta$ and $y = 30t\sin\theta - 5t^2 - 40$. 2

(ii) Find the angle of projection θ. 2

(iii) Find the time taken to strike the frigate. 2

1 (a) $x^2 = 12y$ —— (1)

(i) sub $P(6p, 3p^2)$ into (1)

$\therefore$ LHS $= 36p^2$

RHS $= 12(3p^2) = 36p^2 =$ LHS

$\therefore P(6p, 3p^2)$ lies on the parabola.

(ii) $M_{PQ} = \left(\frac{6p+6q}{2}, \frac{3p^2+3q^2}{2}\right)$

$= \left(3(p+q), \frac{3(p^2+q^2)}{2}\right)$

(iii) $m_{PQ} = \frac{3q^2-3p^2}{6q-6p} = \frac{3(q-p)(q+p)}{6(q-p)}$

$\therefore m_{PQ} = \frac{q+p}{2} \quad (q \neq p)$

But $m_{PQ} = 1 \quad \therefore \frac{q+p}{2} = 1$

$\therefore p+q = 2$

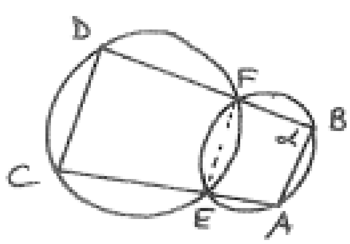
(iv) Now $M_{PQ} = \left(3(p+q), \frac{3(p^2+q^2)}{2}\right)$

$= \left(6, \frac{3(p^2+q^2)}{2}\right)$ using (iii)

when $x=6$, $y=3 \Rightarrow M_{PQ}$ above $y=3$

i.e. locus of midpoints of chords with a gradient of 1 is $x=6$ $(q \neq p)$, $y > 3$.

(b) (i)



Let $\angle FBA = \alpha$

$\therefore \angle FEC = \alpha$ (ext. $\angle$ of cylic quad. = int. opp. $\angle$)

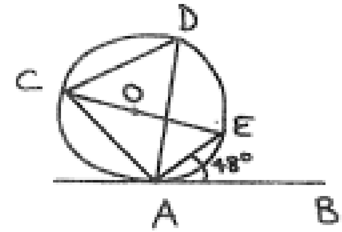
$\therefore \angle CDF = \pi - \alpha$ (opp. $\angle$s in cylic quad are supp.)

But $\angle CDF + \angle ABF = \pi$ (180°)

$\therefore$ int. opp. $\angle$s of quad DBAC

$\Rightarrow AB \parallel CD$.

(c) (i)



$\angle ACE = 48°$ ($\angle$ between tangent and chord at pt. of contact = $\angle$ in the alt. segment)

(ii) $\angle CAE = 90°$ ($\angle$ in semi-circle $= 90°$)

$\therefore \angle CEA = 42°$ ($\angle$ sum of $\Delta = 180°$)

$\therefore \angle ADC = 42°$ ($\angle$s at circum. of circle standing on a common arc are equal.)

2 (a) $\cos 2x + \sqrt{3} \sin 2x = 1$

$\therefore 2\left(\frac{1}{2}\cos 2x + \frac{\sqrt{3}}{2}\sin 2x\right) = 1$

$\therefore \cos(2x - \alpha) = \frac{1}{2}$

where $\cos\alpha = \frac{1}{2}$

$\sin\alpha = \frac{\sqrt{3}}{2}$

$\therefore \tan\alpha = \sqrt{3} \quad \therefore \alpha = \frac{\pi}{3}$

$\therefore \cos\left(2x - \frac{\pi}{3}\right) = \frac{1}{2}$

$\therefore \cos\left(2x - \frac{\pi}{3}\right) = \cos\frac{\pi}{3}$

$\therefore 2x - \frac{\pi}{3} = 2\pi n \pm \frac{\pi}{3}$

$\therefore 2x = 2\pi n \pm \frac{\pi}{3} + \frac{\pi}{3}$

$\therefore 2x = 2\pi n + \frac{2\pi}{3}$ or $2\pi n$

$\therefore x = \pi n + \frac{\pi}{3}$ or πn, where n is any integer

(b) $I = \int \sin^2 6x \, dx$

$= \frac{1}{2}\int 1 - \cos 12x \, dx$

$= \frac{1}{2}\left[x - \frac{\sin 12x}{12}\right] + c$

$\cos 2x = 1 - 2s$

$\therefore \sin^2 x = \frac{1}{2}($

$\therefore \sin^2 6x = \frac{1}{2}(1$

$\int \frac{dx}{x\sqrt{x^2-25}}$

let $x = 5\sec\theta$

$\therefore dx = 5\sec\theta\tan\theta\, d\theta$

$$\therefore I = \int \frac{5\sec\theta\tan\theta\, d\theta}{5\sec\theta\sqrt{25\sec^2\theta-25}}$$

$$= \int \frac{\tan\theta\, d\theta}{5\sqrt{\sec^2\theta-1}}$$

$$= \frac{1}{5}\int \frac{\tan\theta\, d\theta}{\tan\theta}$$

$$= \frac{1}{5}\int 1\, d\theta$$

$$= \frac{1}{5}\theta + c$$

$$= \frac{1}{5}\sec^{-1}\frac{x}{5} + c$$

(d) $I = \int_0^{\frac{1}{2}\ln 3} \frac{e^x}{1+e^{2x}}\, dx$

let $u = e^x$ when $x=0$ $u=1$

$\therefore du = e^x\, dx$ $x=\frac{1}{2}\ln 3$ $u=3^{\frac{1}{2}}$

$$\therefore I = \int_1^{3^{\frac{1}{2}}} \frac{du}{1+u^2}$$

$$= \left[\tan^{-1} u\right]_1^{\sqrt{3}}$$

$$= \tan^{-1}\sqrt{3} - \tan^{-1} 1$$

$$= \frac{\pi}{3} - \frac{\pi}{4}$$

$$= \frac{\pi}{12}$$

3 (a) (i) Step 1: when $n=1$ $(1+x)^n - 1$

$= (1+x)^1 - 1$

$= x$

which is divisible by x

$\therefore$ it is true for $n=1$.

Step 2: Assume it is true for $n=k$ and prove it is true for $n=k+1$

x ... any integer)

$\therefore (1+x)^k = Mx + 1$ ——— (1)

If $n=k+1$ $(1+x)^n - 1 = (1+x)^{k+1} - 1$

$= (1+x)^k(1+x) - 1$

$= (Mx+1)(1+x) - 1$ (sub (1))

$= Mx + Mx^2 + 1 + x - 1$

$= Mx + Mx^2 + x$

$= x(M + Mx + 1)$

which is divisible by x.

$\therefore$ if it is true for $n=k$ so it is true for $n=k+1$.

Step 3: It is true for $n=1$ and so it is true for $n=1+1=2$. It is true for $n=2$ and so it is true for $n=2+1=3$ and so on for all positive integral value of

(ii) $12^n - 4^n - 3^n + 1$

$= 3^n.4^n - 4^n - 3^n + 1$

$= 4^n(3^n-1) - 1(3^n-1)$

$= (3^n-1)(4^n-1)$

$= ((1+2)^n - 1)((1+3)^n - 1)$

divisible by 2 and divisible by 3 using part (i)

$\Rightarrow$ $12^n - 4^n - 3^n + 1$ is divisible by 2 and 3 is 6, for all positive integers $n \geq 1$.

(b) $f(x) = \frac{x}{4-x^2}$

(i) Domain is: all real x except $x = \pm 2$

(ii) $f(x) = \frac{x}{4-x^2}$

$f(-x) = \frac{-x}{4-x^2} = -f(x)$

$\Rightarrow f(x)$ is an odd function.

(iii) $f'(x) = \frac{(4-x^2).1 - x(-2x)}{(4-x^2)^2}$

$= \frac{4+x^2}{(4-x^2)^2} > 0$ for all x except $x =$

its domain.

(iv) For x-intercepts $y=0 \quad \therefore x=0$

For vertical asymptotes $4-x^2=0$

$\therefore x = \pm 2$.

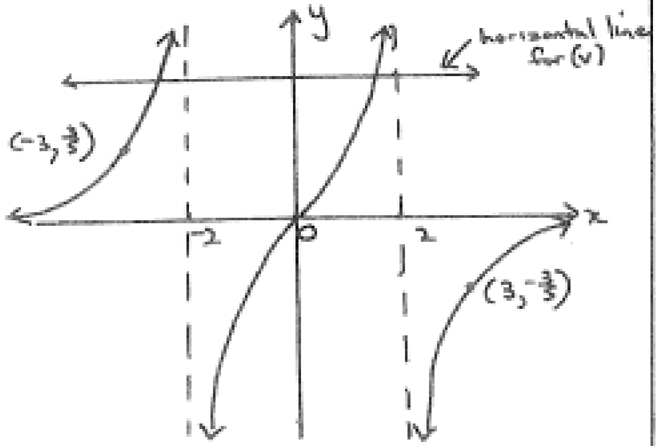
For horizontal asymptotes: $\lim_{x\to\pm\infty} \frac{x}{4-x^2}$

$= \lim_{x\to\pm\infty} \frac{x^2\left(\frac{1}{x}\right)}{x^2\left(\frac{4}{x^2}-1\right)}$

$= \frac{0}{0-1} \left(\begin{matrix}\text{as } x\to\pm\infty \\ \frac{1}{x}, \frac{1}{x^2}\to 0\end{matrix}\right)$

$= 0$

$\therefore$ horiz. asymptote at $y=0$ ($x>2$ or $x<-2$)



(v) As a horizontal line can be drawn above, as shown, to intersect the graph at two distinct points $\Rightarrow$ an inverse function will not exist.

4 (a) Let $P(x) = ax^3+bx^2+cx+d$

As $P(x)$ is monic $\Rightarrow a=1$

$\therefore P(x) = x^3+bx^2+cx+d$

Also $P(0)=-4 \quad \therefore -4=d$

$\therefore P(x) = x^3+bx^2+cx-4$

Also when $P(x)$ is divided by x^2+4 the remainder is $x+8$.

$$\begin{array}{r} x^2+4 \;)\; x^3+bx^2+cx-4 \\ -(x^3 \qquad\quad +4x) \\ \hline bx^2+x(c-4)-4 \\ -(bx^2 \qquad\quad +4b) \\ \hline x(c-4)+(-4-4b) \end{array}$$

But the remainder is $x+8$

$\Rightarrow 1=c-4 \quad \therefore c=5$

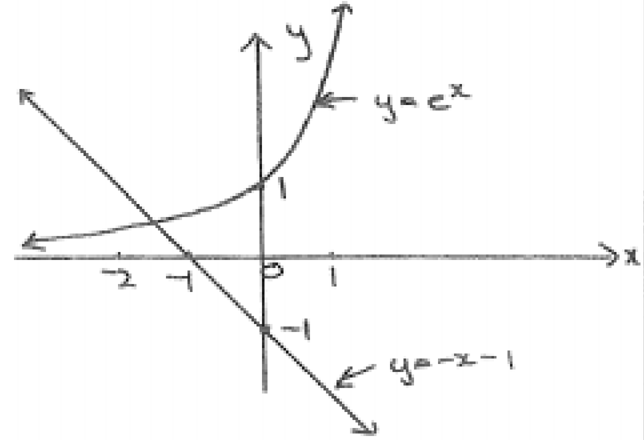
and $8=-4-4b \quad \therefore 4b=-12 \quad \therefore b=-3$

$\therefore P(x) = x^3-3x^2+5x-4$.

(b) (i) If $e^x+x+1=0$

$\therefore e^x = -x-1$

This can be solved graphically by sketching $y=e^x$ against $y=-x-1$.



The sketch indicates that there is only one intersection as shown for $x \doteqdot -1$.

$\Rightarrow e^x+x+1=0$ has only 1 real root and the root is negative.

(ii) By Newton's Method:

$z_2 = z_1 - \frac{P(z_1)}{P'(z_1)}$

Let $P(x) = e^x+x+1$

$\therefore P'(x) = e^x+1$. Let $z_1=-1.5$

$\therefore z_2 = -1.5 - \frac{P(-1.5)}{P'(-1.5)}$

$= -1.5 - \frac{(-0.276869839\ldots)}{1.2231301\ldots}$

$= -1.273638286\ldots$

[As $x \to \infty$, $f(x) \to 1$]

(i) Let $y = 1 + \frac{2}{x-3}$, $x > 3$, $y > 1$

For inverse function interchange x for y

$\therefore x = 1 + \frac{2}{y-3}$

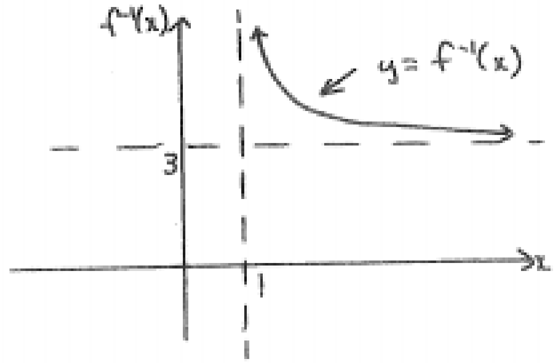
$\therefore x - 1 = \frac{2}{y-3}$

$\therefore \frac{1}{x-1} = \frac{y-3}{2}$

$\therefore y = \frac{2}{x-1} + 3$

$\Rightarrow f^{-1}(x) = 3 + \frac{2}{x-1}$, $x > 1$, $y > 3$.

(ii)



(b) $E = \sin\left[\tan^{-1}\left(\frac{3}{2}\right) + \cos^{-1}\left(\frac{2}{3}\right)\right]$

let $\alpha = \tan^{-1}\frac{3}{2}$, let $\beta = \cos^{-1}\frac{2}{3}$

$\therefore \tan\alpha = \frac{3}{2}$ $\therefore \cos\beta = \frac{2}{3}$

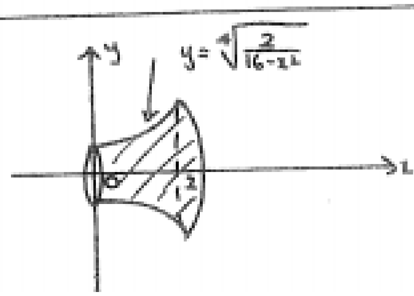
$\therefore E = \sin[\alpha + \beta]$

$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \frac{3}{\sqrt{13}} \cdot \frac{2}{3} + \frac{2}{\sqrt{13}} \cdot \frac{\sqrt{5}}{3}$

$= \frac{6 + 2\sqrt{5}}{3\sqrt{13}}$

(c)



$= \pi \int_0^2 \frac{\sqrt{2}}{\sqrt{16-x^2}}\, dx$

$= \sqrt{2}\pi \left[\sin^{-1}\frac{x}{4}\right]_0^2$

$= \sqrt{2}\pi \left[\sin^{-1}\frac{1}{2} - 0\right]$

$= \sqrt{2}\pi \left(\frac{\pi}{6}\right)$

$= \frac{\sqrt{2}\pi^2}{6}$ units^3

6(a) $\frac{x^2 - 5x}{4 - x} \le -3$ $(x \ne 4)$

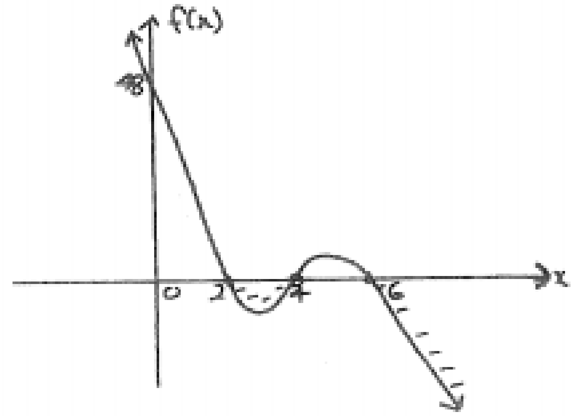
b.c. x $(4-x)^2$ $\therefore (4-x)(x^2-5x) \le -3(4-x)^2$

$\therefore 3(4-x)^2 + (4-x)(x^2-5x) \le 0$

$\therefore (4-x)[3(4-x) + x^2 - 5x] \le 0$

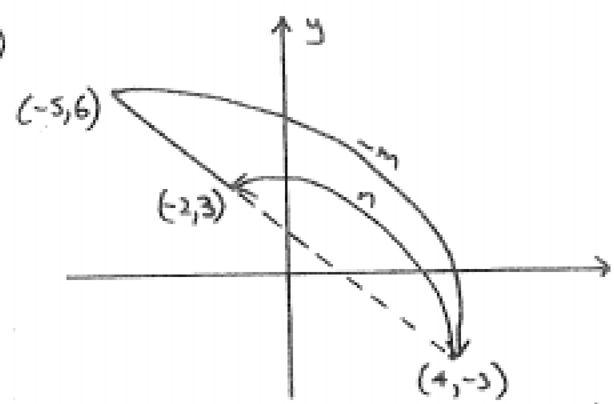
$\therefore (4-x)(x^2 - 8x + 12) \le 0$

$\therefore (4-x)(x-2)(x-6) \le 0$



$\Rightarrow 2 \le x < 4$ or $x \ge 6$

(b)



$(4,-3) = \left(\frac{-m \times -2 + n \times -5}{-m+n}, \frac{-m \times 3 + n \times 6}{-m+n}\right)$

$\therefore -4m + 4n = 2m - 5n$

$\therefore -6m = -9n$

$\therefore \frac{m}{n} = \frac{9}{6} = \frac{3}{2}$

ie $m : n = 3 : 2$

(c) $v^2 = 15 + 2x - x^2$

(i) At end points of motion $v = 0$

$\therefore 15 + 2x - x^2 = 0$

$\therefore x^2 - 2x - 15 = 0$

$(x-5)(x+3) = 0$

$\therefore x = 5$ or -3.

ie end points of motion occur at $x = -3$ and $x = 5$.

(ii) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(\frac{1}{2}[15 + 2x - x^2]\right)$

$= \frac{1}{2}[2 - 2x]$

when $x = -2$ $\quad \ddot{x} = \frac{1}{2}[2+4] = 3$

ie acc'n of particle is $3\ ms^{-2}$ in $\rightarrow$.

7 (a) $\frac{dP}{dt} = k(P - 10000)$ —— ①

(i) $P = 10000 + P_0 e^{kt}$ —— ②

Sub ② into ①:

LHS of ① $= \frac{dP}{dt}$

$= \frac{d}{dt}\left(10000 + P_0 e^{kt}\right)$

$= k P_0 e^{kt}$

$= k(P - 10000)$ (from ②)

$=$ RHS of ①

$\Rightarrow P = 10000 + P_0 e^{kt}$ is a solution of the differential equation.

$\therefore 15000 = 10000 + P_0 e^0$

$\therefore P_0 = 5000$

$\therefore P = 10000 + 5000\, e^{kt}$

When $t = 6$, $P = 25000$

$\therefore 25000 = 10000 + 5000\, e^{6k}$

$\therefore k = \frac{1}{6}\ln 3$.

(iii) Now $P = 10000 + 5000\, e^{(\frac{1}{6}\ln 3)t}$

when $t = 12$, $P = ?$

$\therefore P = 10000 + 5000\, e^{(\frac{1}{6}\ln 3).12}$

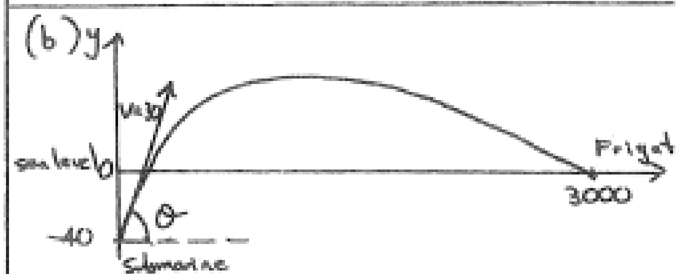
$= 10000 + 5000\, e^{\ln 9}$

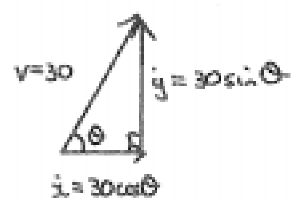
$= 10000 + 45000$

$= 55000$

$\therefore$ After 1 year there are 55 000 tsetse flies.

(b)





(i) Initially $\ddot{x} = 0$, $\ddot{y} = -10$

$\therefore \dot{x} = c_1$, $\dot{y} = -10t + c_2$

when $t = 0$ $\dot{x} = 30\cos\theta$, $\dot{y} = 30\sin\theta$

$\therefore 30\cos\theta = c_1$, $30\sin\theta = c_2$

$\therefore \dot{x} = 30\cos\theta$, $\dot{y} = -10t + 30\sin\theta$

$\therefore x = 30t\cos\theta + c_3$, $y = -5t^2 + 30t\sin\theta + c_4$

when $t = 0$, $x = 0$, $y = -40$

$\therefore c_3 = 0$, $c_4 = -40$

$\therefore x = 30t\cos\theta$, $y = 30t\sin\theta - 5t^2 - 40$ are the parametric equations of motion after $\frac{3\sqrt{3}}{2}$ s.

Now as $\dot{y} = -10t + 30\sin\theta$

$\therefore 0 = \frac{-30\sqrt{3}}{2} + 30\sin\theta$

$\therefore \sin\theta = \frac{\sqrt{3}}{2}$

$\therefore \theta = \frac{\pi}{3}$

ie angle of projection, $\theta = \frac{\pi}{3}$ or $60°$.

(iii) The missile strikes the frigate when $x = 3000$.

Now as $x = 30t\cos\theta$

$\therefore 3000 = 30 \times t \times \cos\frac{\pi}{3}$

$\therefore t = \frac{3000}{30 \times \frac{1}{2}}$

$\therefore t = 200$

$\therefore$ missile strikes the frigate after 200 seconds.